The Sixty-Third Annual William Lowell Putnam Mathematical Competition
Saturday, December 7, 2002

A-1-2002 Let k be a fixed positive integer. The nth derivative of $\dfrac{1}{x^k-1}$ has the form $\dfrac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

Solution: By taking the derivative of the given formula, we find that $P_{n+1}(x) = P_n'(x)(x^k-1) - (n+1)kx^{k-1}P_n(x)$. In particular $P_{n+1}(1) = (-k)(n+1)P_n(1)$. It follows by induction that $P_n(1) = (-k)^n n!$.

A-2-2002 Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Solution: (Karen Collins) Choose two of the points, and take a great circle through these two points. (This can always be done; we can assume one of the points is the north pole, and then we just take a longitude line passing through the other). Of the remaining 3 points, at least two must lie in one of the closed hemispheres determined by the great circle. This gives us 4 points on that closed hemisphere.

A-3-2002 Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \dots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Solution: (Karen Collins and Mark Hovey) $T_n - n$ is the number of such subsets S with at least 2 elements. Given such a subset S, we define a new subset S' as follows. If S contains its average $\mu(S)$, let $S' = S \setminus \{\mu(S)\}$. If S does not contain $\mu(S)$, let $S' = S \cup \{\mu(S)\}$. One can easily check that this defines a bijection between the sets S of the required form that contain their average and the sets that don't. Thus $T_n - n$ is even.

A-4-2002 In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and Player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

Solution: (Karen Collins and Mark Hovey) Player 0 wins. This is basically a case by case analysis. Since the determinant is invariant up to sign under permutations of rows and columns, we can assume player 1 puts a 1 in the $(1,1)$ position. The player 0 puts a 0 in the $(2,2)$ position. At this point, we can still take the transpose of the matrix, so we can assume player 1 puts a 1 somewhere below or on the main diagonal. This gives us four cases. In case player 1 puts a 1 in the $(3,3)$ position, player 0 puts a 0 in the $(3,2)$ position. This forces player 1 to put a 1 in the $(1,2)$ position to avoid a 0 column. Player 0 then puts a 0 in the $(2,1)$ position, forcing player 1 to put a 1 in the $(2,3)$ position to avoid a 0 row. Player 0 then puts a 0 in the $(3,1)$ position, giving two identical rows. The other cases are similar.

A-5-2002 Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every rational number appears in the set

$$\left\{\frac{a_{n-1}}{a_n} : n \geq 1\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots\right\}.$$

Solution: (Mark Hovey and Dave Rusin) Define the height of a rational number m/n in lowest terms to be the larger of m and n. We prove every rational number is in the set by induction on the height. The base case is 1, which is certainly in there. Now suppose we have m/n. If the height is m, then

$$m/n = (m-1)/n + 1.$$

Since $(m-1)/n$ has smaller height, it is a_k/a_{k+1} for some k. This means that $m/n = a_{2k+2}/a_{2k+3}$.

Now we want to use $b_{2k+1} = a_{2k+1}/a_{2k+2}$. One can easily check that $b_{2k+1} = \frac{b_k}{b_k+1}$, or, said another way, that

$$b_k = \frac{b_{2k+1}}{1 - b_{2k+1}}.$$

So now suppose that the height of m/n is n, so that $m/n < 1$. Then we form

$$\frac{m/n}{1-(m/n)} = \frac{m}{n-m}.$$

This has strictly smaller height, so must be b_k for some k. It follows that m/n is b_{2k+1}. This completes the induction.

A-6-2002 Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

Solution: (Dave Rusin) The answer is that the series converges only if $b = 2$. To see that it does not converge for $b > 2$, let S_N denote the partial sum of the first N terms. We consider the case where $N = b^D - 1$, so all of the numbers $n \leq N$ have D digits or fewer. Collect the terms by number of digits. This gives

$$S_N = \sum_{d=1}^{D} \sum_{n=b^{d-1}}^{b^d-1} \frac{1}{f(n)} = \sum_{d=1}^{D} \frac{1}{f(d)} \sum_{n=b^{d-1}}^{b^d-1} \frac{1}{n}$$
$$> \sum_{d=1}^{D} \frac{1}{f(d)} \int_{b^{d-1}}^{b^d} \frac{1}{x}\, dx = (\log b) S_D.$$

If the sum converges to S, then taking the limit as D goes to infinity gives us

$$S \geq (\log b) S.$$

This is a contradiction if $b > 2$ (since S is certainly positive).

We must now prove the series does converge if $b = 2$. This is proven similarly, using lower sums instead of upper sums in the integral approximation above. This gives $S_{2^D-1} < A + \log(2) S_D$. Using this one can check that $S_N < A(\sum_{k=0}^{\infty} (\log 2)^k)$ for some A, and that is enough to prove the series converges.

B-1-2002 Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

Solution: The answer is all outcomes between 1 and 99 shots made are equally likely, so the probability is $\frac{1}{99}$. Prove by induction that the probability of hitting k shots out of n is $\frac{1}{n-1}$ if $1 \leq k < n$.

B-2-2002 Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

Solution: Suppose there is a face F with at least four sides. Player 1 signs that face. No matter what player 2 does, player 1 then signs a face adjacent to F with no vertices on F in common with where player 2 signed. There are now 2 faces player 1 can sign to win, and player 2 can only block one of those, so player 1 wins in 3 moves.

If all the faces have 3 sides, then one can easily see from the formula $v - e + f = 2$ that the polyhedron is a tetrahedron, so does not have enough faces.

B-3-2002 Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - (1 - \frac{1}{n})^n < \frac{1}{ne}.$$

Solution: The inequality

$$\frac{1}{e} - (1 - \frac{1}{n})^n < \frac{1}{ne}$$

is equivalent to

$$n \ln(\frac{n-1}{n}) > \ln(\frac{n-1}{n}) - 1$$

which is obvious. The other inequality

$$\frac{1}{2ne} < \frac{1}{e} - (1 - \frac{1}{n})^n$$

is equivalent to

$$n \ln(\frac{n-1}{n}) < \ln(\frac{2n-1}{2n}) - 1.$$

To prove this, we in fact show that

$$n \ln(\frac{n-1}{n}) < \ln(\frac{2n-2}{2n}) - 1.$$

This is equivalent to showing that

$$(n-1) \ln(\frac{n-1}{n}) < \ln 2 - 1,$$

or

$$(1 - \frac{1}{n})^{n-1} < \frac{2}{e}.$$

We show this by showing that the function $(1 - \frac{1}{x})^{x-1}$ is decreasing. Since the inequality holds for $n = 2$, it holds for all $n > 2$.

This takes some effort with the derivative; to show the first derivative is negative, you end up having to check another inequality. This one is also accomplished by taking a derivative.

B-4-2002 An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than 2/3.

Solution: Let $P_o(n)$ denote the probabillity of winning in an odd number of guesses when we start with n numbers and $P_e(n)$ denote the probability of winning in an even number of guesses. Note first that there is a strategy with $P_e(3) = 2/3$; just make your first guess be 2.

Lemma: If there is a strategy with $P_e(n) = 2/3$, then there is a strategy with $P_e(n+3) = 2/3$.

Proof: Make your first guess be $n+2$, and your second guess be $n+1$ or $n+3$, whichever is legal. After that follow the strategy that has $P_e(n) = 2/3$. Then you will win in an even number of guesses if the chosen number is $n+1$, $n+3$, or one of the $(2/3)n$ winning possibilities of the first strategy. Thus you win for $(2/3)n + 2 = (2/3)(n+3)$ possibilities, proving the lemma.

As a consequence of this lemma, there is a strategy with $P_e(n) = 2/3$ for every n that is a multiple of 3. Now 2002 is congruent to 1 mod 3, so complete the proof with the following lemma.

Lemma: Suppose there is a strategy with $P_e(n) = 2/3$. Then there is a strategy with $P_o(n+4) > 2/3$.

Proof: Make your first guess be $n+1$. If the chosen number is greater than $n+1$, make your second guess be $n+3$. Otherwise follow the give strategy for n. You will win if the number is $n+1$, $n+2$, $n+4$, or any of the $(2/3)n$ winning possibilities for n. Thus you win for $(2/3)n + 3 > (2/3)(n+4)$ possibilities, completing the proof.

B-5-2002 A palindrome in base b is a positive integer whose base-b digits read the same backwards and forwards; for example, 2002 is a 4-digit palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base b for at least 2002 different values of b.

Solution: (Dave Rusin) We will show that in fact there is an n is $1a1$ in base b for arbitrarily many values of b. Note that this happens if and only if $n-1 = b(b+a)$, so whenever we can factor $n-1 = xy$ for positive integers x, y with $x \le y < 2x$, n will be a 3-digit palindrome in base x. We do this as follows. Let L be a fixed large positive integer, and let P denote the set of primes between L and $2L$. We let

$$n - 1 = \prod_{p \in P} p(p+1).$$

Then, for each subset of P we get a factorization $n - 1 = xy$, where

$$x = \prod_{p \in S} p \prod_{p \in P \setminus S} (p+1), \text{ and } y = \prod_{p \in P \setminus S} p \prod_{p \in S} (p+1).$$

All of these factorizations are distinct, since the prime factors of $p+1$ are all $\leq L$. We claim that each of these factorizations will do the job. Indeed, it suffices to check that the ratio x/y is between $1/2$ and 2. The largest value V of this ratio is when S is empty, and the smallest value of this ratio is when $S = P$ and is the reciprocal of V. So it suffices to show that $V \leq 2$. But

$$\begin{aligned}\log V = \sum_{p \in P} \log(p+1) - \log p \leq \sum_{k=L}^{2L-1} \log(k+1) - \log k \\ = \log 2L - \log L = \log 2 < 2.\end{aligned}$$

Thus all these factorizations give us bases for which n is a palindrome. Obviously by taking L large enough we can make P as large as we want, completing the proof.

B-6-2002 Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p.)

Solution: This is one where knowing some algebra helps. Work modulo p. Suppose there is a nontrivial relation $ax + by + cz = 0$, with a, b, c integers in $\mathbb{Z}/p$. Then

$$(ax + by + cz)^p = a^p x^p + b^p y^p + c^p z^p = ax^p + by^p + cz^p = 0$$

Similarly, $ax^{p^2} + by^{p^2} + cz^{p^2} = 0$. This means the rows of the matrix are linearly dependent, and so the determinant is 0.

This proves that the polynomial $ax+by+cz$ divides the determinant of this matrix over $\mathbb{Z}/p$. In fact, the determinant is

$$x \prod_{a \in \mathbb{Z}/p} (y - ax) \prod_{b,c \in \mathbb{Z}/p} (z - by - cx).$$

Indeed, every one of the factors divides the determinant, and they are all relatively prime, so their product divides the determinant. The determinant and this product are both homogeneous of total degree $p^2 + p + 1$, so they must agree up to a multiple in $\mathbb{Z}/p$. But this multiple has to be 1 by looking at the $xy^p z^{p^2}$ term.